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CONSERVATIVE WEIGHTINGS AND EAR-DECOMPOSITIONS OF GRAPHS

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A subset J of edges of a connected undirected graph $G=(V,E)$ is called a *join* if $|C\cap J|\leq |C|/2$ for every circuit C of G. Answering a question of P. Solé and Th. Zaslavsky, we derive a min-max formula for the maximum cardinality μ of a join of G. Namely, $\mu=(\varphi+|V|-1)/2$ where φ denotes the minimum number of edges whose contraction leaves a factor-critical graph.

To study these parameters we introduce a new decomposition of G, interesting for its own sake, whose building blocks are factor-critical graphs and matching-covered bipartite graphs. We prove that the length of such a decomposition is always φ and show how an optimal join can be constructed as the union of perfect matchings in the building blocks. The proof relies on the Gallai-Edmonds structure theorem and gives rise to a polynomial time algorithm to construct the optima in question.

1. Introduction

This paper is concerned with a problem related to ear-decompositions, matchings, and T-joins of an undirected graph. For a general account on the topic we refer to the book of Lovász and Plummer [9]. The topic is also related to coding theory. (See, for example, [11]).

Let $G=(V,E)$ be a finite undirected graph. By an *ear-decomposition* of G we mean a sequence $G_0,G_1,\dots,G_t=G$ of subgraphs of G where G_0 consists of one node and no edge, and each G_i arises from G_{i-1} by adding a path P_i for which the two (not-necessarily distinct) end-nodes belong to G_{i-1} while the inner nodes of P_i do not. The paths P_i are called *ears*. Note that P_i may consist of a single edge and such an ear is called *trivial*. Sometimes we say the set $\mathcal{P}=\{P_1,P_2,\dots,P_t\}$ of paths is an ear-decomposition. The *length* of a path or ear is the number of its edges. An ear is *odd* if its length is odd. An ear-decomposition is called *odd* if each of its ears is odd.

It is well-known that G has an ear-decomposition if and only if G is 2-edge-connected. Actually, an ear-decomposition of any 2-edge-connected subgraph of G can be continued to become an ear-decomposition of G. In particular, any circuit of G can be chosen to be the first member of an ear-decomposition.

It is clear that the number of ears of an ear-decomposition is independent of the decomposition and equals $|E|-|V|+1$.

There are some known results on ear-decompositions [9]. A basic one is due to L. Lovász [7]. To formulate it we recall that a graph is called *factor-critical* (in

AMS subject classification code (1991): 05 C 70, 05 C 75, 94 B 60 short, *critical*) if $G-v$ has a perfect matching for every node v of G. (Critical graphs are sometimes called *hypomatchable*.)

Theorem 1.1. [7] *A graph G is factor-critical if and only if G possesses an odd ear-decomposition. Furthermore, for any edge e of a critical graph G there is an odd ear-decomposition of G such that the first ear uses e.*

The second statement is not explicitely stated in Lovász' paper but it follows immediately from his proof. (It is not true, though, that any odd circuit can be the first ear of an odd ear-decomposition.) One of our purposes is to answer the question of how far a graph is from being critical. More precisely:

What is the minimum number $\varphi=\varphi(G)$ of even ears in an ear-decomposition of G?

We call an ear-decomposition of G *optimal* if the number of even ears is $\varphi(G)$. Lovász' theorem can be formulated this way: given a graph G, $\varphi(G)=0$ if and only if G is critical. Let us call a subset F of edges *critical-making* if contracting the elements of F leaves a critical graph. We will prove (Theorem 3.2) that φ can be interpreted as the minimum cardinality of a critical-making set. (One way is trivial: pick up one edge from each even ear of an optimal ear-decomposition. Clearly this set of φ edges is critical making. The other direction, as we shall see, is not difficult either.)

There are some results in the literature, besides Lovász' theorem, concerning this second interpretation of φ. For example, Hetyei [6], [9: Theorem 5.4.1] proved that $\varphi(G)=1$ when G is matching-covered (that is, every edge of G belongs to a perfect matching). Another result (see Theorem 2.2 below) asserts that for a bipartite graph G $\varphi(G)=1$ if and only if G is matching-covered.

We are going to establish a good characterization for general graphs having $\varphi(G)=1$ (Corollary 4.10). A more general purpose of the paper is to provide a min-max expression for φ. In order to formulate it we need a another parameter of G that was introduced by P. Solé and T. Zaslavsky [15].

A subset J of edges of an undirected graph $G=(V,E)$ is called a *join* if $|C\cap J|\leq |C|/2$ holds for every circuit C of G. P. Solé and T. Zaslavsky posed the problem of studying the maximum cardinality $\mu(G)$ of a join of G. Originally, Solé and Zaslavsky investigated μ in a different context. They pointed out that $\mu(G)$ is equal to the so-called covering radius of the cycle code of G. The investigation of covering radius was originated by MacWilliams and Sloane in their book [11: page 173, Research Problem (6.1)]. (We shall not discuss these concepts further here. Interested readers should consult [15]). In Section 3 we will introduce some other equivalent definitions of $\mu(G)$.

We are going to prove the following relationship between φ and μ.

Main Theorem. $\varphi(G)=2\mu(G)-|V|+1$ *for any 2-edge-connected graph $G=(V,E)$.*

This is a min-max theorem and, as usual, proving that max $\leq$ min is rather easy (Lemma 4.2). The proof of the other direction will be based on a theorem asserting that a non-critical graph G always has a certain subgraph H (called a strong end) such that $\varphi(G)\leq\varphi(G/H)+1$ and $\mu(G)\geq\mu(G/H)+|V(H)|/2$. By combining these inequalities and the induction hypothesis on G/H, a proof of the main theorem will follow.

Beside ear-decompositions, we will introduce another decomposition of G. An *end-decomposition*, to be defined precisely in the next section, is one whose building blocks are critical graphs and matching-covered bipartite graphs. We will prove that the length of any end-decomposition is always φ and show how an optimal join can be constructed as the union of perfect matchings in the building blocks.

For $U \subseteq V$ subset $E(U)$ denotes the set of edges of G with both end-nodes in U. $G(U) := (U, E(U))$ is called a *subgraph induced by* U. Given an edge $e = xy$ of graph $G = (V, E)$, by the *contraction* of e we mean a graph $G/e = (V', E')$ where V' arises from V by identifying x and y into a new node z while $E' = \{ab : ab \in E, \{a,b\} \cap \{x,y\} \neq \emptyset\} \cup \{az : ax \in E, a \neq y\} \cup \{az : ay \in E, a \neq x\}$. There is a 1-1 correspondence between the edge-set E' of G' and $E - e$ and sometimes we do not distinguish between a subset of E' and the corresponding subset of E.

For a connected induced subgraph $(U, E(U))$ of G the *contraction* G/U denotes a graph arising from G by contracting all the edges in $E(U)$. Sometimes we refer to G/U as a graph arising from G by *shrinking* U into one node. Any edge e' of G/U corresponds to an edge e of G with at least one end-node in $V - U$. We will not distinguish between e' and e. In particular, for any matching M of G/U the corresponding edge set in G forms a matching of G and we will denote this latter matching with the same letter M.

The *deletion* $G - U$ means the subgraph of G induced by $V - U$ (that is, $G - U = G(V - U)$).

For a subset X of edges and weighting $w : E \to \mathbb{R}$ we use the notation $w(X) := \sum(w(e) : e \in X)$.

2. Matchings and decompositions

Let us cite first some important notions and theorems from matching theory (see [9]). For a subset X of nodes of a graph G let $\Gamma(X) := \{v \in V - X :$ there is an edge $uv \in E$ with $u \in X\}$ and let $q(X)$ be the number of odd components in $G - X$.

By a *matching* M we mean a subgraph of G with no two edges incident. A *perfect matching* is one covering all the nodes of G. A graph G having a perfect matching is called *perfectly matchable* (or, in short, *matchable*). We call a matching M of a graph a *near-perfect matching* if M leaves precisely one node exposed.

Let us recall Tutte's theorem: A graph has a perfect matching if and only if the so-called *Tutte condition* holds: $q(X) \leq |X|$ for each subset X of nodes. The *deficiency* $def(G)$ of a graph G is defined to be $\max(q(X) - |X| : X \subseteq V)$. A set X with $q(X) - |X| = def(G)$ is called a *barrier* of G. Obviously, any matching of G leaves at least $def(G)$ nodes exposed and the Berge-Tutte formula, a slight generalization of Tutte's theorem, asserts that any matching of maximum cardinality leaves precisely $def(G)$ nodes exposed. It follows that a graph is critical if and only if the empty set is its unique barrier.

Let S be a barrier of G. Construct a bipartite graph G_S as follows. Delete the even components of $G - S$, delete the edges induced by S, and contract each odd component of $G - S$. We call G_S the *bipartite graph associated with barrier* S.

Suppose that G has no perfect matching. Let $D(G)$ denote the set of nodes exposed by at least one maximum matching and let $A(G):=\Gamma(D(G))$. The following fundamental result is called the Gallai - Edmonds structure theorem.

Theorem 2.1. [1,4] *$A(G)$ is a barrier and $D(G)$ is the union of odd components of $G-A(G)$. $A(G)$ can be described as the unique barrier X for which the union of odd components of $G-X$ is of minimum cardinality. Moreover, the odd components in $D(G)$ are critical, and for any non-empty subset X of $A(G)$ the number of odd components in $D(G)$ having a neighbour in X is larger than $|X|$.* ∎

Note that Edmonds' matching algorithm [1] computes not only a maximum matching but the sets $A(G)$ and $D(G)$, as well. The core of Edmonds' algorithm is an augmenting step, we call it *Edmonds-augmentation*, that starts with a matching and either finds a bigger matching or finds $A(G)$ and $D(G)$. Note that, with careful implementation, one Edmonds augmentation can be carried out in $O(|E|)$ steps.

Let G be perfectly matchable. G is *matching-covered* if every edge belongs to a perfect matching. G is *elementary* if the union of perfect matchings forms a connected spanning subgraph of G. We have already introduced the concept of critical graphs along with Lovász' theorem. In our investigations elementary bipartite graphs play an equally important role.

Let $G=(S,T;E)$ be a (2-edge-connected) bipartite graph with a perfect matching M. For $s\in S$ let $s'\in T$ denote the node for which $ss'\in M$. Construct a directed graph $D:=D(G;M)$ on node-set S by letting uv $(u,v\in S)$ be an edge of D if $u'v$ is an edge of G.

Theorem 2.2. [9: Theorems 4.1.1 and 4.1.6] *For a (2-edge-connected) bipartite graph $G=(S,T;E)$ with a perfect matching M the following are equivalent.*

a. *G is elementary,*
b. *G is matching covered,*
c. *G has an ear-decomposition such that only the first ear P is even (and, furthermore, P can be chosen to contain any specified edge f of G),*
d. *$|\Gamma(X)|>|X|$ For every proper subset $\emptyset\neq X\subset S$,*
e. *For a perfect matching M the w_M-distance of any two nodes is non-positive,*
f. *$D(G;M)$ is strongly connected.* ∎

In particular, if G consists of parallel edges between two nodes, then G is elementary biparite.

We will also need some new concepts concerning matchings. For a perfectly matchable graph H we call a barrier X a *strong barrier* if $H-X$ consists of $|X|$ odd components and no even components, each odd component is critical, and the bipartite graph H_X associated with X is elementary. We call a perfectly matchable graph *half-elementary* if it has a strong barrier. The name is justified by the following theorem.

Theorem 2.3. [9: Theorem 5.2.2] *In an elementary graph G define two nodes x,y to be related if either $x=y$ or $G-\{x,y\}$ is not matchable. This relation is an equivalence relation and each equivalence class is a strong barrier.* ∎

For a graph $G=(V,E)$ let $H=G(U)$ be a subgraph induced by $U\subseteq V$ so that H is half-elementary with strong barrier X. We call H a *strong end* of *G attached* *at* X if X is a cut-set of G separating $U-X$ from $V-U$ or if $U=V$ (in which case G itself is half-elementary). A matching M of G is said to *fit* a strong end $G(U)$ if the restriction of M to U forms a perfect matching of $G(U)$.

(Note that X need not be a barrier of G and $G-U$ may not be connected. Also, when we say that a strong end H of G is attached at X, this will automatically mean that X is a strong barrier of H.)

For example, if $G:=G_T$ is a *doubled tree*, that is, G arises from a tree T by replacing each edge by two parallel edges, then there is a one-to-one correspondence between the pendant edges of T and the strong ends of G_T.

If a graph G includes a strong end attached at X, then $G-x$ cannot have a perfect matching for $x\in X$ since in $G-x$, the set $X-x$ violates the Tutte condition. Hence critial graphs have no strong ends. The converse is also true:

Theorem 2.4. *A non-critical graph $G=(V,E)$ has a strong end $G(U)$ and a maximum matching M fitting $G(U)$.*

Proof. If G has no perfect matching, define a subset $S:=A(G)$. S is non-empty since G is not critical. If G is perfectly matchable, define $S:=A(G-v)+v$ where v is an arbitrarily chosen node of G. By Theorem 2.1, in both cases S is a non-empty barrier of G such that $G-S$ contains at least $|S|$ critical components.

Let $G_S:=(S,T;E_1)$ be the bipartite graph associated with barrier S. By Theorem 2.1 any maximum matching of G matches the elements of S with distinct odd components of $G-S$. Hence there exists a matching M_S of G_S covering S. For every $s\in S$ let s' denote the node for which $ss'\in M_S$.

Construct a directed graph $D:=D(G_S;M_S)$ on node-set S by letting uv ($u,v\in S$) be an edge of D if $u'v$ is an edge of G_S. By contracting each maximal strongly connected component of D we obtain a digraph $\bar{D}$ which is acyclic. Therefore $\bar{D}$ has a sink-node $\bar{x}$. This sink-node $\bar{x}$ corresponds to a set $X:=\{x_1,x_2\dots,x_k\}\subseteq S$ that induces a strongly connected component of D so that no edge of D leaves X. (Note that by a depth-first search computation set X can be computed in $O(|E_S|)$ time.) Let $X':=\{x'_1,\dots,x'_k\}$ and let $D_1,\dots,D_k$ be the critical components of $G-S$ corresponding to the elements of X'.

Since X induces a strongly connected component of digraph D, Theorem 2.2 (parts (a) and (f)) implies that the subgraph of G_S induced by $X\cup X'$ is elementary. Since there is no edge in D leaving X, there is no edge in G connecting $D_1\cup\dots\cup D_k$ and $S-X$. Hence $D_1,\dots,D_k$ are components of $G-X$, as well, and therefore $X\cup D_1\cup\dots\cup D_k$ induces a strong end of G attached at X. ∎

Let $H=G(U)$ be a strong end of a graph $G=(V,E)$. By an *end-reduction at* U we mean the contraction G/U. By an *end-decomposition* of $G_0:=G$ we mean a sequence $(G_0,U_0),(G_1,U_1),\dots,(G_k,U_k)$ where each U_i induces a strong end of G_i, each G_{i+1} arises from G_i by an end-reduction at U_i, G_k is critical and U_k is empty. The total number k of end-reductions is called the *length* of the end-decomposition. We say that a sequence of matchings $M_0,M_1,\dots,M_k$ *fits* the end-decomposition if M_i is a maximum matching of G_i ($i=0,\dots,k$) and each M_i ($i=0,1,\dots,k-1$) fits $G_i(U_i)$.

For example, if G_T is a doubled tree, then an end-reduction corresponds to the contraction of a pendant edge of T. In this case the length of any end-decomposition is the same, namely, the number of edges of T. It will turn out that, for every graph G, the length of any end-decomposition of G is the same, namely, $\varphi(G)$.

Repeated applications of Theorem 2.4 yields:

Corollary 2.5. *Every graph has an end-decomposition* $(G_0,U_0),(G_1,U_1),\ldots,(G_k,U_k)$ *and a fitting sequence of matchings* $M_0,M_1,\ldots,M_k$.

Let the restriction of M_i to $G_i(U_i)$ $(i=1,\ldots,k-1)$ be denoted by N_i and $N:=N_0\cup\ldots\cup N_{k-1}\cup M_k$. In the next section we will prove that N forms an optimal join of G while an optimal ear-decomposition of G can be obtained by appropriately composing optimal ear-decompositions of $G_0(U_0),\ldots,G_{k-1}(U_{k-1})$ and G_k. Therefore, in order to construct an optimal join and an optimal ear-decomposition of G we will have to be able to construct these sequences.

The proof of Theorem 2.4 shows how to compute the first members $G_0 = G, U_0, M_0$ of these three sequences. It requires the use of at most $|V|/2$ Edmonds-augmentations along with $O(|E_S|)$ additional work. By repeated applications of this step the complete sequences G_i, U_i, M_i can be computed. A naive implementation shows that at most $k|V|/2 \leq |V|^2/2$ Edmonds-augmentations plus $O(|E|)$ additional work are required.

We can, however, improve on this estimation by observing that M_{i+1} can be computed from M_i by using at most two Edmonds-augmentations $(i=1,\ldots,k)$. To see this we assume, without loss of generality, that $i=0$. Let M_0 be a maximum matching of $G_=G$, N_0 the restriction of M_0 to U_0 and let $M_1' := M_0 - N_0$. Recall that N_0 is a perfect matching in $G_0(U_0)$. Let u_0 denote the node of G_1 arisen by the contraction of U_0.

We claim that the cardinality of M_1' is at most one less than that of a maximum matching of G_1. Indeed, if there is a matching R of G_1 with $|R| = |M_1'|+2$, then by leaving out the element of R incident to u_0 (if there is such) we obtain a matching R' of G_1 avoiding u_1 with $|R'| = |M_1'|+1$. But then $R' \cup N_0$ would be a matching of G larger than M_0, a contradiction. By this claim a maximum matching M_1 of G_1 and set $A(G_1)$ can be computed by using at most two Edmonds-augmentations (one to increase M_1', the second to find $A(G_1)$).

Summing up, the sequences G_i, U_i, M_i belonging to an end-decomposition of G can be computed by at most $5|V|/2$ Edmonds-augmentations plus $O(|E|)$ additional steps. Hence the overall complexity of the algorithm can be bounded by $O(|V||E|)$.

3. Equivalent definitions for μ and φ

Let us provide another interpretation of μ. A ± 1 edge-weighting $w: E \to \{1,-1\}$ of G is called *conservative* if $w(C) \geq 0$ for every circuit C of G (that is, there is no circuit of negative total weight). For a ± 1 weighting w let J_w denote the set of negative edges. For a subset J of edges let w_J be defined by $w_J(e) := -1$ if $e \in J$ and $w_J(e) := +1$ if $e \in E - J$.

Clearly, for a conservative weighting w, J_w forms a join and hence $\mu(G)$ is equal to the maximum number of negative edges in a conservative weighting of G. Conversely, if J is a join, then w_J is a conservative weighting. We call a conservative weighting (and also a join) realizing μ *optimal*.

Conservative ± 1 weightings are intimately related to T-joins. Let T be a subset of V with even cardinality. A subset J of edges is called a *T-join* if a node v is incident to an odd number of edges from J precisely when v belongs to T. Note that any subset J of edges is a T_J-join where T_J is defined to be the subset of nodes with an odd number of incident edges from J.

Since a subset J of edges is a join if and only if J is a minimum T_J-join, we have yet another meaning of μ: it is the largest cardinality of a minimum T-join over all even subsets T of nodes.

There is an extensive literature on T-joins and T-cuts [2, 3, 8, 11, 12, 13]. Here we mention only an easy but basic result, due to A.Sebő.

Theorem 3.1. [13] *Let w be a conservative weighting of a graph $G=(V,E)$ and T a subset of nodes of even cardinality. A T-join J is of minimum w-weight if and only if w' is conservative where $w'(e):=-w(e)$ when $e\in J$ and $w'(e):=w(e)$ when $e\in E-J$. In particular, if J is either a circuit of zero w-weight (case of $|T|=0$) or a path of minimum w-weight, connecting two specified nodes of G (case of $|T|=2$), then the operation of changing the sign of each $w(e)$, $e\in J$, results in a conservative weighting.*

Proof. The symmetric difference $C\otimes J$ of a circuit C and a T-join J is a T-join. Therefore if there is a circuit C of negative w'-weight, then $w(J\otimes C)=w(J)-w(J\cap C)+w(C-J)=w(J)+w'(C)<w(J)$, showing that J is not a minimum w-weight T-join.

Conversely, suppose that there is no circuit of negative w'-weight. For a T-join J', $D:=J\otimes J'$ is a union of edge-disjoint circuits. Hence $w'(D)\geq 0$ and thus $w(J')=w(J)-w(J-J')+w(J'-J)=w(J)+w'(D)\geq w(J)$, showing that J is a minimum w-weight T-join. ∎

Specializing Theorem 1.2 to $w\equiv 1$ one obtains a result of Mei-gu Guan [5] that serves as a bridge between minimum cardinality T-joins and conservative weightings: a T-join J is of minimum cardinality if and only if there is no circuit of negative total w_J-weight. This means that a minimum cardinality T-join can be determined if one is able to decide whether there is a negative circuit with respect to a ± 1 weighting. The other direction is true, as well. That is, if one wants to decide whether a given ± 1 weighting w is conservative, one can do it by determining the minimum cardinality of a T_J-join where J is the set of negative edges.

We will need the following interpretation of φ.

Theorem 3.2. *For a 2-edge-connected graph G $\varphi=\varphi(G)$ is equal to the minimum cardinality of a critical-making edge-set.*

Proof. Let us consider an ear-decomposition $\mathcal{P}$ of G with φ even ears. Choose one edge from each even ear. By contracting these φ edges $\mathcal{P}$ transforms into an odd ear-decomposition of the contracted graph G'. By Theorem 1.1 G' is critical, demonstrating the existence of a critical-making set of φ elements.

Conversely, assume that there is a critical making set of k elements. In order to show that $\varphi(G)\leq k$ we use induction on k. The inequality is true for $k=0$ by Theorem 1.1. Let $e=uv$ be one of the k contracted edges and let G' denote the graph obtained from G by contracting e. (The loop arising from e is left out.) For every edge $x\neq e$ of G we let x' denote the corresponding edge of G'. Denote by

v_e the contracted node. Then G' can be made critical by contracting $k-1$ edges and hence, by induction, G' has an ear-decomposition $\mathcal{P}' = \{P'_1, P'_2, \ldots, P'_t\}$ using at most $k-1$ even ears.

Let P'_h be the first ear in $\mathcal{P}'$ containing an edge f' incident to v_e. f' corresponds to an edge f of G that is incident to u or v, say to u. Let P'_j be the first ear in $\mathcal{P}'$ that contains an edge of G' corresponding to an edge of G incident to v. Clearly $j \geq h$.

For $P'_i \in \mathcal{P}' - \{P'_j\}$ let P_i denote the set of edges of G corresponding to the edges of P'_i. For P'_j let P_j denote the union of $\{e\}$ and the set of edges corresponding to the edges of P'_j. It is easily seen that $\mathcal{P} := \{P_1, \ldots, P_t\}$ is an ear-decomposition of G. Furthermore, $\mathcal{P}$ contains at most one more even ear than $\mathcal{P}'$ (namely, if P'_j is odd, then P_j is even). Therefore $\varphi(G) \leq \varphi(G') + 1 = k$, as required. ∎

Remark. Originally, we defined φ only for 2-edge-connected graphs. By this theorem the definition of $\varphi(G)$ can be extended for every connected graph G. Namely, let $\varphi(G)$ be defined as the minimum cardinality of a critical making edge-set. This is clearly the same as the minimum number of even ears of an ear-decomposition of G' where G' arises from G by replacing each cut-edge e by two parallel edges e' and e''. Since in any ear-decomposition of G' the circuit formed by e' and e'' must appear as an even ear we see that $\varphi(G)$ is equal to the number of cut-edges of G plus $\sum \varphi(G_i)$ where the sum is taken over the maximal 2-edge-connected subgraphs G_i of G.

The following statement will be used later.

Lemma 3.3. *For any edge f of a 2-edge-connected graph G there is an optimal ear-decomposition of G such that the first ear uses f.*

Proof. By Theorem 1.1 the lemma is true for critical graphs. Assume that G is not critical, that is, $\varphi(G) \geq 1$. Let us consider an ear-decomposition of G containing $\varphi(G)$ even ears. Choose an edge e from one of these even ears that is distinct from f. Let G' denote the graph obtained from G by contracting e. By the preceding theorem $\varphi(G') = \varphi(G) - 1$. By induction G' has an optimal ear-decomposition such that the first ear uses f. By the second part of the proof of the preceding theorem this ear-decomposition defines an optimal ear-decomposition of G. ∎

Remark. It is not true that for every edge f there is an optimal ear-decomposition that starts with an even ear using f. Indeed, take G to be K_4 minus an edge and let f be the edge of G connecting the two nodes of degree three. It can be proved (but we do not need it here) that for a given edge f there is an opimal ear-decomposition in which the first ear is even and uses f if and only if $\varphi(G) > \varphi(G/f)$.

4. Joins and ears

The purpose of this section is to prove the main theorem formulated in the Introduction. Along the way we prove some other results concerning the behaviour of these parameters in an end-decomposition of G.

Lemma 4.1. *Let $G=(V,E)$ be a graph obtained from $G'=(V',E')$ by adding an ear P of length p. Then $\mu(G')\geq\mu(G)-\lfloor p/2\rfloor$.*

Proof. Let u and v be the two (not necessarily distinct) end-nodes of P. Let w be an optimal ±1 weighting of G. Let p^- (p^+) denote the number of negative (positive) edges of P and let w' denote the restriction of w to E'. First suppose that $p^-\leq p^+$. Then $p^-\leq\lfloor p/2\rfloor$ and we have $\mu(G')\geq\mu(G)-p^-\geq\mu(G)-\lfloor p/2\rfloor$, as required. Second, suppose that $p^->p^+$, that is, $w(P)<0$. Then, since w is conservative, every path in G' connecting u and v has w'-weight at least $-w(P)$, a positive number. Theorem 3.1 shows that interchanging the sign of w' along a path of minimum w'-weight provides a conservative weighting w'' of G' and the number of edges with negative w''-weight is at least $\mu(G)-p^--w(P)$. Therefore $\mu(G')\geq\mu(G)-p^--w(P)=\mu(G)-p^+\geq\mu(G)-\lfloor p/2\rfloor$ as required. ∎

Lemma 4.2.

$$\varphi(G)\geq 2\mu(G)-|V|+1. \tag{4.1}$$

Proof. By induction on the number of ears ($=|E|-|V|+1$). Let P be the last ear of an ear-decomposition of G with $\varphi(G)$ even ears. Let ε be 1 if P is odd and 0 if P is even. Using Lemma 4.1 and the induction hypothesis we have $\varphi(G)=\varphi(G')+1-\varepsilon\geq 2\mu(G')-|V'|+1-\varepsilon+1\geq 2(\mu(G)-\lfloor p/2\rfloor)-|V'|-\varepsilon+2=2\mu(G)-(p-1+|V'|)+1=2\mu(G)-|V|+1$. ∎

The next theorem determines φ and μ for critical graphs and for half-elementary graphs.

Theorem 4.3. *(a) For a critical graph $G=(V,E)$ $\varphi(G)=0$ and any near-perfect matching of G is a maximum join, that is, $\mu(G)=(|V|-1)/2$. (b) Let H be a half-elementary graph with strong barrier X and f an edge connecting X and $V(H)-X$. Then any perfect matching of H is a maximum join, that is, $\mu(H)=|V(H)|/2$. Furthermore, $\varphi(H)=1$ and there is an optimal ear-decomposition starting with an even ear that contains f.*

Proof. (a) If G is critical, then $\varphi(G)=0$ and by Lemma 4.2 $\mu(G)\leq(|V|-1)/2$. Moreover a near-perfect matching M of G is a join of $(|V|-1)/2$ elements and hence $\mu(G)=(|V|-1)/2$, that is, M is an optimal join.

(b) Let X be a strong barrier of H. Since H has an even number of nodes, $\varphi(H)\geq1$. To see that equality actually holds we can suppose that X induces no edge since deleting these edges leaves H half-elementary. We use induction on the number of nodes. If every component of $H-X$ is a singleton, then H is elementary bipartite and, by Theorem 2.2, $\varphi(H)=1$. Moreover, the first ear in an ear-decomposition of H is even and uses f.

Now let K be a component of $H-X$ with $|K|>1$. Let H' denote the graph obtained from H by shrinking K into a new node v_K. H' is half-elementary as X is a strong barrier for this graph. By induction, H' has an ear-decomposition $\mathcal{P}'$ so that the first ear is even while all the other ears are odd. Let P be the first ear in $\mathcal{P}'$ that contains v_K. Suppose that x_1v_1 and x_2v_2 are edges in H corresponding to the two edges of P incident to v_K ($x_1,x_2\in X$ and $v_1,v_2\in K$). If $v_1=v_2$, then by inserting $\mathcal{P}_k$ into $\mathcal{P}'$ right after P we obtain an ear-decomposition of H in which the first ear is even and uses f while all the other ears are odd. In particular, $\varphi(H) = 1$ follows.

Suppose now that $v_1 \neq v_2$. Introduce a new edge $e = v_1v_2$ in K. By Theorem 1.1 there is an odd ear-decomposition $\mathcal{P}_K$ of $K+e$ so that the first ear P_0 contains e. P_0 is an odd circuit and hence $P_0 - e$ is an even path in K connecting v_1 and v_2. Define P' by replacing the subpath (x_1v_K, v_Kx_2) by $(x_1v_1, P_0 - e, v_2x_2)$. Obviously the length of P' has the same parity as that of P. Modify $\mathcal{P}'$ by replacing first P by P' and, then, by inserting $\mathcal{P}_K - \{P_0\}$ right after P'. This way we obtain an ear-decomposition of H in which the first ear is even and uses f while all the other ears are odd. In particular, $\varphi(H) = 1$ follows.

By Lemma 4.2 $\mu(H) \leq |V(H)|/2$. Hence we have equality as any perfect matching is a join of cardinality $|V(H)|/2$. ∎

Lemma 4.4. *Let $H=(U,E(U))$ be a strong end of G attached at X and let $G' := G/U = (V',E')$. Then*

$$\mu(G) \geq \mu(G') + |V(H)|/2 \text{ and} \tag{4.2}$$

$$\varphi(G) \leq \varphi(G') + 1. \tag{4.3}$$

Proof. Let u' denote the node of G' arisen by shrinking U into one node. Let J' be an optimal join of G' and let J denote the subset of edges of G corresponding to J'. Let N be any perfect matching of H. We claim that $J \cup N$ is a join of G. Indeed, let C be any circuit of G. If C is disjoint from X then it is either a circuit of H or corresponds to a circuit of G' avoiding u'. In both cases we have $w_{J \cup N(C)} \geq 0$ as N is a join of H and J' is a join of G'. If C intersects X, then C partitions into edge-disjoint paths so that the ends of these paths belong to X while the inner nodes (if any) do not. Therefore the $w_{J \cup N}$-weight of each such path is non-negative and thus $w_{J \cup N(C)} \geq 0$. Hence $J \cup N$ is a join of G of cardinality $\mu(G') + |V(H)|/2$ and (4.2) follows.

By Theorem 4.3 H has an ear-decomposition $\mathcal{P}_1$ using one even ear. By Lemma 3.3 G' has an optimal ear-decomposition $\mathcal{P}'$ such that the first ear contains u'. Any ear P' in $\mathcal{P}'$ for which one or both end-nodes are u' corresponds to a path P'' of G having one or both end-nodes in X, respectively. Let $\mathcal{P}''$ be obtained from $\mathcal{P}'$ by replacing all such P' by P''. Then $\mathcal{P} := (\mathcal{P}_1, \mathcal{P}'')$ is an ear-decomposition of G using $\varphi(G') + 1$ even ears. Therefore $\varphi(G) \leq \varphi(G') + 1$. ∎

We are now in the position to prove the main theorem:

Theorem 4.5. *For every connected graph $G = (V,E)$*

$$\varphi(G) = 2\mu(G) - |V| + 1. \tag{4.4}$$

Proof. If G contains a cut-edge e, then for $G' := G/e = (V',E')$ one has $\varphi(G') = \varphi(G) - 1, \mu(G') = \mu(G) - 1, |V'| = |V| - 1$. Hence (4.4) follows by induction. Hence we may assume that G is 2-edge-connected.

If G is critical, then $\varphi(G) = 0$ and (4.4) is equivalent to $\mu(G) = (|V| - 1)/2$. This has already been proved in Theorem 4.3.

If G is not critical, then, by Theorem 2.2, G has a strong end $H=(U,E(U))$. Let $G'=G/U$. By induction we assume that (4.4) holds for G', that is, $\varphi(G')=2\mu(G')-|V'|+1$.

Combining this relation with (4.1), (4.3), and (4.2) we get

$$\begin{aligned}(4.5)\quad \varphi(G)\leq\varphi(G')+1 &= 2\mu(G')-|V'|+2\leq 2\mu(G)-(|V(H)|+|V'|)+2\\ &= 2\mu(G)-|V|+1\leq\varphi(G).\end{aligned}$$

Hence equality holds throughout from which (4.4) follows. ∎

For 2-edge-connected graphs there is an equivalent formulation of the main theorem.

Theorem 4.5'. *The maximum number of odd ears in an ear-decomposition of a 2-edge-connected graph $G=(V,E)$ is equal to the minimum of $w(E)$ over all conservative ±1 weightings w of G.*

Proof. The maximum in question is $|E|-|V|+1-\varphi(G)$ since the total number of ears in any ear-decomposition is $|E|-|V|+1$. On the other hand the minimum of $w(E)$ can be expressed as $|E|-2\mu(G)$. By (4.4) the theorem follows. ∎

Our next result descibes how μ and φ behave under the operation of end-reduction.

Theorem 4.6. *Let $H=(U,E(U))$ be a strong end of G attached at X and f an arbitrary edge of H connecting X and $U-X$. Let $G':=G/U=(V',E')$ be the end-reduction of G at U. Then*

(a) *$\varphi(G)=\varphi(G')+1$. There is an optimal ear-decomposition of G that starts with an optimal ear-decomposition of H. There is an optimal critical-making edge-set of G that consists of an optimal critcal-making edge-set of G' plus edge f.*

(b) *$\mu(G)=\mu(G')+|U|/2$ and there is a maximum join of G which is the union of a maximum join of G' and a perfect matching of H. Every maximum join of G arises as the union of a maximum join of H and a maximum join of G'.*

Proof. (a) By (4.5) we have equality in (4.3). Hence the ear-decomposition $\mathcal{P}$ given in the proof of Lemma 4.4 has $\varphi(G)$ even ears. This ear-decomposition may start with any optimal ear-decomposition of H. By Theorem 4.3 (b) H has an optimal ear-decomposition in which the first ear is even and uses a specified edge f. By Theorem 3.2 it follows that an optimal critical-making edge-set can be obtained from an optimal ear-decomposition by chosing an arbitrary edge from each even ear. This implies the last statement of part (a).

(b) By (4.5) we have equality in (4.3) and in (4.2). Therefore $J\cup N$ constructed in the proof of Lemma 4.4 is a maximum join of G.

Let now J be an arbitrary maximum join of G and $\mathcal{P}$ an optimal ear-decomposition of G that starts with an (optimal) ear-decomposition of H. By the estimation in the proof of Lemma 4.2, $|J\cap P|=\lfloor |P|/2\rfloor$ for each $P\in\mathcal{P}$. Hence J_k is an optimal join of G_k where G_k is the union of the first k ears of $\mathcal{P}$ and J_k is the restriction of J to G_k. In particular, $J\cap H$ is a maximum join of H. Hence there is a path P of H connecting x and y for which $|P\cap J|\geq|P-J|$ for every pair x,y of nodes of H. Therefore $J':=J-H$ is a join of G'. ∎

In Corollary 2.5 we proved that a graph G has an end-decomposition. The next theorem tells us how any end-decomposition of G determines an optimal join and an optimal critical making edge-set.

Theorem 4.7. *Let $(G_0,U_0),(G_1,U_1),\ldots,(G_k,U_k)$ be an arbitrary end-decomposition of a 2-edge-connected graph $G=(V,E)$. Then the length k of the end-decomposition depends only on G and equals $\varphi(G)$.*

(a) *Let f_i be an arbitrary edge of G_i $(i=0,\ldots,k-1)$ connecting X_i and $V(G_i)-X_i$ (where X_i denotes the strong barrier belonging to $G_i(U_i)$). Then $\{f_0,f_1,\ldots,f_k-1\}$ is an optimal critical-making edge-set of G.*

(b) *Let N_i be a any perfect matching of $G_i(U_i)$ $(i=0,\ldots,k-1)$ and M_k any near-perfect matching of G_k. Then $J:=N_0\cup\ldots\cup N_{k-1}\cup M_k$ is an optimal join of G.*

Proof. Repeated applications of Theorem 4.6 implies the result. ∎

Algorithmic remarks. In Section 2 we described how to compute an end-decomposition of G along with a fitting sequence of matchings in $O(|V||E|)$ steps. By the preceding theorem an optimal join and an optimal critical-making set can be immediately constructed. Since we proved that in (4.3) always equality holds the ear-decomposition of G described in the proof of Lemma 4.4b is an optimal ear-decomposition of G. In the next section we will see that for bipartite graphs a significantly simpler approach is available.

Theorem 4.5 provides a characterization for graphs with $\varphi(G)=1$. For this special case we have a more structured characterization. The theorems above imply that if $\varphi(G)\geq 1$, then there is an optimal ear-decomposition starting with an even circuit. If $\varphi(G)=1$, then the first ear is even and the other ears are odd. Obviously, such a graph has a perfect matching. Therefore in any characterization for graphs with $\varphi(G)=1$ we may assume that G has a perfect matching.

Corollary 4.8. *Let M be a perfect matching of a graph $G=(V,E)$. The following are equivalent:*

(a) $\varphi(G)=1$,

(b) *M is a maximum join,*

(c) *the w_M-distance of any two nodes is non-positive.*

(d) There are no two disjoint non-empty subsets A,B of nodes so that $G-(A\cup B)$ contains $|A|+|B|$ odd components (i.e., $A\cup B$ is a barrier) among which $|A|$ components are connected only to A and the other $|B|$ components are connected only to B.

Proof. (a)→(b). By (4.1) we have $\mu(G)=|V|/2$, that is, M is a maximum join.

(b)→(c). Immediate from Theorem 3.1.

(c)→(d). If there were two subsets A,B with the given properties, then the w_M-distance of any node of A and any node of B would be positive.

(d)→(a). G has a perfect matching, so $|V(G)|$ is even and hence $\varphi(G)\geq 1$. By Theorem 4.7 it suffices to show that there is an end-decomposition of G of length 1. By Theorem 2.4 G has a strong end $H=(U,E(U))$. Assume that H is attached at its strong barrier A. Let $G':=G/U=(V',E')$ and let u' denote the contracted node. We are going to show that G' is critical.

Indeed, if G' is not critical, then $B:=A(G')$ is non-empty. Since the restriction M' of M to G' is a near-perfect matching of G', M' is a maximum matching of G'

avoiding u'. By Theorem 2.1 we see that $u' \notin B$, that is A and B are disjoint and satisfy the properties in (d), a contradiction. ∎

Remark. (For readers interested in coding theory.) From (4.1) it follows that the covering radius $cr(G)(=\mu(G))$ of the cycle space of a connected graph $G=(V,E)$ is always at least $\lfloor n/2 \rfloor$ where $n=|V|$. For odd n $cr(G)$ is precisely $\lfloor n/2 \rfloor$ if and only if G is critical. For even n Corollary 4.8 describes the graphs for which $cr(G) = n/2$. The next result may also have some interest for coding theorists.

Corollary 4.9. *If $G=(V,E)$ is a connected subgraph of $G'=(V,E')$, then $\mu(G) \geq \mu(G')$.*

Proof. Immediate from Theorem 4.5. ∎

5. Bipartite graphs

In this section we show that for a bipartite graph G value $\mu(G)$ is related to another interesting parameter of G. We need some further notions. In a directed graph the set of edges entering a subset X of nodes called a *directed cut* if no edge leaves X. A digraph is *strongly connected* if there is a directed path from every node to any other. It is well-known that a digraph is strongly-connected if and only if it does not contain directed cuts. Another easy but important result claims that a directed graph is strongly connected if and only if it has a so-called directed ear-decomposition where each ear is a directed path.

A fundamental theorem of Lucchesi and Younger concerning directed cuts is as follows.

Theorem 5.1. [10] *In a digraph the maximum number of disjoint directed cuts is equal to the minimum number of edges covering all the directed cuts.*

(A set F of edges is said to *cover* a directed cut C if $C \cap F \neq \emptyset$.) It is easy to see that the minimum can be interpreted as the minimum number of edges whose contraction leaves a strongly connected digraph.

For a graph G let $c(X)$ denote the number of components of $G-X$. Let $G=(A,B;E)$ be a 2-edge-connected bipartite graph and let G' denote a directed graph obtained from G by orienting each edge from A to B.

Theorem 5.2. [3] *The minimum number $\sigma=\sigma(G)$ of edges entering A in a strongly connected orientation of G is equal to $\max \sum c(A_i)$ where the maximum is taken over all partitions $\{A_i\}$ of A.* ∎

We call a strongly-connected orientation of G *optimal* if the number of edges entering A is minimum. Obviously a set F of edges entering A in a strongly-connected orientation of G corresponds to a set of edges in G' covering all directed cuts of G'. Furthermore any partition $\{A_i\}$ of A determines $\sum c(A_i)$ disjoint directed cuts. Therefore $\sigma(G)$ can be viewed as the minimum number of edges covering all directed cuts of G' and Theorem 5.2 can be considered as a refinement of the Lucchesi-Younger theorem for directed bipartite graphs.

We need the following version:

Theorem 5.2'. *Let F be an edge-set of minimum cardinality ($\sigma(G)$) entering A in a strongly connected orientation of G. Then there is a partition $\{A_1,\dots,A_k\}$ of A such that for each component K of $G-A_i$ there is exactly one edge from F connecting A_i and K ($i=1,\dots,k$).* ∎

The main purpose of this section is to establish the following:

Theorem 5.3. *For a bipartite graph $G=(A,B;E)$ $\sigma(G)=\mu(G)$, that is, the maximum cardinality of a join of G is equal to the minimum number of edges entering A in a strongly connected orientation of G.*

For the proof we need the following basic result of P.Seymour.

Theorem 5.4. [13] *A subset F of edges of a bipartite graph G is a join if and only if there are $|F|$ disjoint cuts of G such that each contains one edge from F.* ∎

In [3] we provided a refinement of Seymour's theorem.

Theorem 5.5. *A subset F of edges of a bipartite graph $G=(A,B;E)$ is a join if and only if there is a partition $\mathcal{P}$ of A such that for any $P\in\mathcal{P}$ each component of $G-P$ has at most one entering edge of F.* ∎

Proof of Theorem 5.3. Let F be a set defined in Theorem 5.2'. Theorems 5.2' and 5.5 show that F is a join and hence $\mu(G)\geq\sigma(G)$.

To see the reverse inequality let F be a maximum join. Let $\mathcal{P}$ be a partition guaranteed in Theorem 5.5. Then $\mu(G)=|F|\leq\sum(c(X):X\in\mathcal{P})\leq\sigma(G)$. Here the second inequality follows from (the trivial part of) Theorem 5.2. ∎

Next, we exhibit a further result concerning σ. As an application of the Lucchesi-Younger theorem D. Younger proved that the maximum number of disjoint cuts in an (undirected) bipartite graph G is the maximum number of disjoint directed cuts in G'. This result and its proof were communicated to me by W. Pulleyblank and, as far as I know, it has never been published. Younger's proof consists of showing how a family of k disjoint (undirected) cuts of G can be transformed into a family of k disjoint directed cuts of G'.

In the present context Younger's theorem can be stated as follows.

Theorem 5.6. *In a bipartite graph G the maximum number M of disjoint cuts is $\sigma(G)$.*

Proof. Theorem 5.2 shows that there are $\sigma(G)$ disjoint cuts in G, that is, $M\geq\sigma(G)$. To see the other direction let us consider a family of M disjoint cuts. Let F be a set of edges containing one edge from each of the given M cuts. By Theorem 5.4 F is a join. By Theorem 5.3 we have $M=|F|\leq\mu(G)=\sigma(G)$, as required. ∎

Finally we show a relationship between optimal ear-decompositions and optimal strongly-connected orientations of a bipartite graph G.

On one hand, every optimal ear-decomposition determines an optimal strongly-connected orientation as follows. If P is an ear in the decomposition, then orient its edges so as to form a directed path that, in addition, goes from A to B whenever P is odd. This way we get a strongly-connected orientation of G in which the number

$\varrho(A)$ of edges entering A is $(|E|-\pi(G))/2$ where $\pi(G)$ denotes the number of odd ears. Since $\pi(G)+\varphi(G)=|E|-|V(G)|+1$, Theorem 4.7 shows that $\sigma(G)\leq\varrho(A)=(|E|-\pi(G))/2=(\varphi(G)+|V(G)|-1)/2=\mu(G)=\sigma(G)$,that is the given orientation is optimal.

On the other hand, let us be given an optimal strongly connected orientation of G (with $\sigma(G)$ edges entering A). Then it has a directed ear-decomposition. We claim that this decomposition, when the ears are considered undirected, is optimal. Indeed, let φ' and π' denote the number of even and odd ears, respectively. Then we have $\sigma(G)=\varrho(A)\geq(|E|-\pi')/2=(\varphi'+|V(G)|-1)/2\geq(\varphi(G)+|V(G)|-1)/2=\mu(G)=\sigma(G)$, from which equality follows everywhere. In particular, $\varphi'=\varphi(G)$, as required.

6. Concluding remarks and open problems

In this paper we analysed some properties of parmeters φ and μ. Parameter φ has various interpretations. By definition φ is the minimum number of even ears in an ear-decomposition of G. It is equal to the minimum cardinality of a critical-making set of edges. Furthermore, φ is the number of end-reductions in any end-decomposition of G.

Similarly, μ has various meanings. By definition μ is the maximum cardinality of a join. Furthermore, μ is the maximum number of negative edges in a conservative ±1 weighting, μ is the maximum cardinality of a minimum T-join where the maximum is taken over all even T, and μ is the covering radius of the cycle code of a graph.

We derived a min-max formula concerning μ and φ, namely $\mu=(|V|-1+\varphi)/2$. Also, we analysed the relationship of these parameters to end-decompositions of G. Concerning these parameters several questions arise. Given a weight-function $w: E\to R_+$, what is the maximum weight of a join? What is the minimum weight of a critical-making set? When w is a $0-1$ function these questions specialize to the following. Given a subset F of edges, find a join $J\subseteq F$ of maximum cardinality. When does there exist a critical-making subset of F? If there is one, find the smallest. How can all optimal critical-making sets be described? And the same question for maximum joins.

Another type of question arises if we vary the notion of a join. For example, D. Welsh asked if there is a polynomial time algorithm for the maximum cardinality $\mu^<$ of a strong join. A *strong join* J of a graph G is a subset of edges so that $|C\cap J|<|C|/2$ holds for every circuit C of G. Recently A. Fraenkel and M. Loebl [17] proved that this problem is NP-complete even for planar bipartite graphs. On the other hand in [18] we proved that $\mu^<$ is at most $\lfloor(|V|-1)/2\rfloor$ for every 2-edge-connected graph and for bipartite graphs we described a polynomial-time algorithm to decide if $\mu^<=\lfloor(|V|-1)/2\rfloor$.

It is natural to consider analogous problems for directed graphs $D=(V,E)$. Namely, find a conservative ±1 weighting of D so that the number of negative edges is maximum, or, equivalently, find a maximum join of D. (A weighting is called *conservative* if there is no directed circuit of negative total weight. A subset J of edges is a *join* if $|C\cap J|\leq|C|/2$ for every directed circuit.) Note that the problem of deciding whether a specified ± 1 weighting is conservative or not is much easier in a directed graph than in an undirected graph. In this light the following result is slightly surprising.

Theorem 6.1. *The maximum join problem for digraphs is NP-complete.*

Proof. We reduce the problem to the undirected max-cut problem that is known to be NP-complete. The reduction runs as follows.

From network flow theory it is well-known that an edge-weighting w of a digraph is conservative if and only if there is a node-function $\pi : V \rightarrow \mathbb{R}$ such that $w(uv) \geq \pi(v) - \pi(u)$ for every edge uv of D. Therefore an optimal ± 1 weighting can be obtained as follows. Take a partition $\{V_1, V_2, \ldots, V_k\}$ of V into non-empty subsets and call an edge uv a *forward* edge if $u \in V_i$ and $v \in V_j$ with $i < j$. Define $w(uv)$ to be -1 on forward edges and $w(uv) := 1$ otherwise. Therefore the problem is equivalent to (*) finding a partition $\{V_1, V_2, \ldots, V_k\}$ of V so that the number of forward edges is maximum. Let $\vec{\mu}(D)$ denote this maximum.

Now let $G = (V, E)$ be an undirected graph and let $\kappa(G)$ denote the maximum cardinality of a cut of G. Construct a digraph $D = (V, A)$ from G by raplacing each edge uv of G by two oppositely directed edges uv and vu. Clearly, $\vec{\mu}(D) \geq \kappa(G)$. We claim that $\vec{\mu}(D) \leq \kappa(G)$. Indeed, let $\{V_1, V_2, \ldots, V_k\}$ $(k \geq 2)$ be a partition of V with $\vec{\mu}(D)$forward edges of D. Then the bipartition $(V_1 \cup V_3 \cup \ldots \cup V_{2i+1} \cup \ldots, V_2 \cup V_4 \cup \ldots \cup V_{2i} \cup \ldots)$ of G determines a cut of G with $\vec{\mu}(D)$ edges. Hence $\vec{\mu}(D) = \kappa(G)$ showing that problem (*) is not easier than the NP-complete undirected max-cut problem. ∎

Note added in proof. In a recent paper entitled "On Lovász' cathedral theorem", Z. Szigeti solved the problem of minimum weight critical-making sets by proving that the family of edge-sets F, whose contraction decreases φ by $|F|$, forms the independent sets of a matroid.

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